Table 2 Comparison of the tip deflection V

D	By Timoshenko theory with the shear coefficient equal to 0.666	0.870	Present work (second-order approximation)
2	1.24844	1.19037	1.34094
4	1.06211	1.04759	1.08751
6	1.02760	1.02115	1.03927
8	1.01553	1.01190	1.02205
10	1.00994	1.00761	1.01413
20	1.00248	1.00190	1.00354
40	1.00062	1.00046	1.00089
100	1.00010	1.00008	1.00014

and

$$G(\partial w/\partial y + dv/dz) = 0 \text{ at free edges} \tag{15}$$

Following the procedure of previous section, Timoshenko equation can be shown to be a special case of this formulation and the second-order approximation equation can be obtained as

$$\begin{gathered} GAv_s'' + q = 0 \\ E(Iv_b^{iv} + b_2\phi_2''') = q \\ E(b_2v_b''' + B_{22}\phi_2'') + GC_{22}\phi_2 = 0 \end{gathered} \tag{16}$$

Results and Discussion

Two simple examples are considered in order to bring out the necessity of the present formulation in the short column range, 1) stability of a simply-supported rectangular column, and 2) cantilever rectangular beam subjected to an end load. The cross-sectional constants involved in second-order approximation equations can be evaluated as[4]

$$\begin{gathered} A = bd \qquad I = (\tfrac{1}{12})bd^3 \\ b_2 = -C_{22} = (\tfrac{1}{120})bd^5 \qquad B_{22} = (\tfrac{17}{20160})bd^7 \end{gathered} \tag{17}$$

By using second-order approximation equations one can obtain the critical load parameter of a simply-supported rectangular column as

$$\lambda = 1/[1 + k^2\pi^2/12D^{*2} + 84k^2\pi^2/(k^2\pi^2 + 10D^{*2})] \tag{18}$$

whereas Timoshenko theory yields

$$\lambda = 1/[1 + (k^2\pi^2/12\beta D^{*2})] \tag{19}$$

and the value of λ is unity by Euler's theory. Comparison of results obtained by various theories is shown in Table 1.

In the case of a cantilever rectangular beam subjected to a load P_t at the free end, the nondimensional deflection at the free end V by second-order approximation equations is

$$V = \left[\frac{\cosh\mu}{\cosh\mu + 84}\right]\left[1 + \frac{(252}{\mu^2)}\left(1 - \frac{\tanh\mu}{\mu}\right)\right]$$

with $\mu = [(840G/E)D^2]^{1/2}$

Timoshenko theory gives

$$V = 1 + (E/4\beta GD^2)$$

and the value is unity by elementary theory.

Comparison of results obtained by various theories is given in Table 2. In both the cases the results indicate that use of refined shear coefficient in Timoshenko equation increases the difference in the results from the more accurate results by the second-order approximation equations.

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An Efficient Triangular Shell Element

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Various investigators have analyzed arbitrary shells by using curved elements. Gallagher[1] has presented an exhaustive review of available plate and shell elements. Though the development of membrane elements is almost complete, the existing bending elements may still be replaced by more efficient formulations. Oden and Wempner[2] suggested to use the linear shear theory and to neglect the shear energy. A discrete equivalent of the Kirchhoff assumptions is then introduced over the element in order to relate rotations and the transverse displacement. Recently, a family of triangular shell elements, based on the discrete Kirchhoff assumptions, has been presented by the present author.[3]

In this study, a new triangular shell element is developed by using the linear shear theory. The shear energy is neglected and a discrete version of the Kirchhoff assumptions is introduced over the element.

1. Formulation of the Element

The true geometry $z(x,y)$ of the triangular element has been approximated by a shallow quadratic surface as described by Bonnes et al.[4] The strain-displacement relations for a thin shallow element are defined as follows:

$$\epsilon = e_m + \zeta\kappa \tag{1a}$$

$$\begin{bmatrix} e_{m1} \\ e_{m2} \\ e_{m12} \end{bmatrix} = \begin{bmatrix} u_{,x} - z_{,xx}w \\ v_{,y} - z_{,yy}w \\ u_{,y} + v_{,x} - 2z_{,xy}w \end{bmatrix} = e_m \tag{1b}$$

$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix} = \begin{bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{bmatrix} = \kappa \tag{1c}$$

where u, v, and w are displacements along ζ_1, ζ_2, and ζ_3 directions of a right-hand orthogonal coordinate system. β_x and β_y are rotations of the normal ζ_3 along ζ_1 and ζ_2 directions, respectively, ζ is the thickness coordinate of the element such that $-h/2 < \zeta < h/2$, and h is the thickness of the element.

By definition, the membrane forces and the bending moments are

$$\begin{aligned} (N_xN_yN_{xy}) &= he_m{}^TD \\ (M_xM_yM_{xy}) &= (h^3/12)\kappa^TD \end{aligned} \tag{2}$$

where the superscript T indicates the transpose of a vector or a matrix

$$D = E/(1-\nu^2)\begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & (1-\nu)/2 \end{bmatrix}$$

E is equal to the modulus of elasticity of the material, and ν is equal to Poisson's ratio of the material.

The internal energy of the element is defined by

$$U = \tfrac{1}{2}\int e_m{}^T De_m dv + \tfrac{1}{2}\int \kappa^T D\kappa\zeta^2 dv \tag{3}$$

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Table 1 Vertical displacement of the cylinder (in.)

Mesh	This study	Ref. 3	Ref. 4
2×3	3.507	3.965	2.65
4×5	3.750	3.797	3.56
8×12	3.733	3.730	3.716

The displacements u, v, and w are each approximated by a 9-term cubic polynomial over the element

$$u = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8y^3 + a_9x^2y \quad (4a)$$

The displacement v and w are each represented by a similar polynomial. The element axes x,y are, in general, assumed to be arbitrarily oriented.

It should be noted that as a result of the discrete Kirchhoff assumptions, the bending properties are invariant under rotation of the axes. For the shell element, one may represent u, v, and w displacements by complete cubic polynomials and eliminate the extra degrees of freedom by the potential energy method. For simplicity, the 9-term cubic polynomials are used in this study without violating the compatibility requirements. This element is capable to give accurate results for arbitrarily shaped and arbitrarily oriented triangular meshes. The rotations β_x and β_y are each approximated by a complete quadratic polynomial

$$\beta_x = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 \quad (4b)$$

The rotation β_y is represented by a similar polynomial.

The element thus defined has 39 degrees of freedom, 11 assigned with each corner node ($u\ u_{,x}\ u_{,y}\ v\ v_{,x}\ v_{,y}\ w\ w_{,x} w_{,y} \beta_x \beta_y$) and 2 with each midpoint node ($\beta_x \beta_y$).

The stiffness matrix of the element is obtained in a routine manner by minimizing the internal energy U as defined by Eq. (3). A discrete version of the Kirchhoff assumptions is introduced as follows:

$$\text{At each corner node, } w_{,x} = -\beta_x \text{ and } w_{,y} = -\beta_y \quad (5a)$$

$$\text{At each midpoint node, } w_{,t} = -\beta_t \quad (5b)$$

The normal rotation β_n at the midpoint node of each side is eliminated by equating it to half the sum of the normal rotations at the bounding nodes of that side. There are 12 constraints introduced over the element. This will reduce the 39 degrees of freedom of the element to 27, that is, 9 degrees of freedom assigned with each corner node ($u, u_{,x}\ u_{,y}\ v\ v_{,x}\ v_{,y}\ w\ \beta_x\ \beta_y$).

2. Illustrative Examples

The bending behavior of the element has been tested by analyzing a number of plate problems. The results have been independently reported by Stricklin et al.[5] and by the present author.[6] In the following shell problems, the distributed loads are replaced by a set of concentrated loads. The element stiffness matrices are obtained by a 16-point numerical integration formula for coarse mesh sizes and by a 9-point formula for finer mesh sizes.

a) Circular cylinder

The shell is an open circular cylinder, loaded by its own weight, supported on rollers along the curved edges and free along the straight edges (Fig. 1). Only $\frac{1}{4}$ of the shell is studied because of the double symmetry. The results are given in Table 1 and Fig. 1. These are compared with those reported by other investigators.[3,4,7]

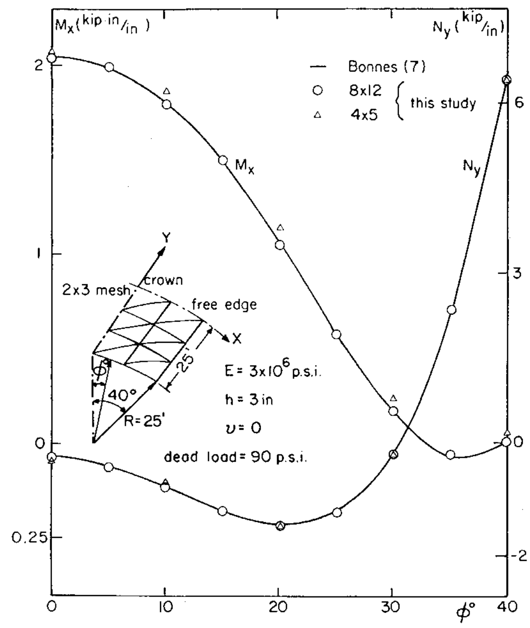

Fig. 1 Circular cylinder: moment M_x and force N_y along $y = 0$.

b) Hyperbolic paraboloid

A square hypar with straight clamped edges subject to uniform normal pressure is studied for 4×4, 6×6, and 8×8 meshes (Fig. 2). The results are shown in Table 2 and Fig. 2. These are compared with those reported by other investigators.[3,7,8]

3. Conclusion

Based on the results reported in Ref. 6 and herein, it may be concluded that the discrete Kirchhoff triangular elements

Table 2 Middle point displacement of the hypar (in. $\times 10^{-3}$)

Mesh	This study	Ref. 3	Ref. 7	Ref. 8
4×4	8.824	8.940	...	...
6×6	8.885	8.780	...	...
8×8	8.888	8.680	...	8.708
10×10	...	...	8.945	...

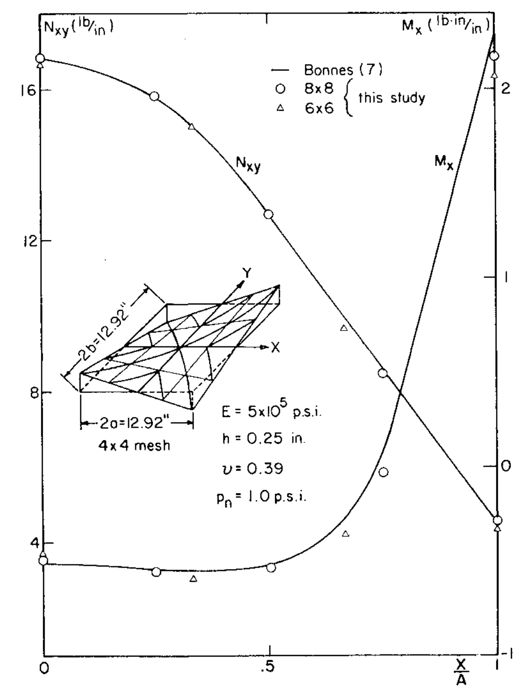

Fig. 2 Clamped hypar: moment M_x and force N_{xy} along $y = 0$.

are relatively more efficient and precise than the existing Kirchhoff-type triangular elements. The nonlinear stability behaviour of shells is presently under investigation with the use of this element. The results obtained so far are excellent.

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Buckling of Orthotropic Annular Plates

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Introduction

The buckling of orthotropic circular plates, due to in-plane compressive loads, has been investigated by Woinowski-Krieger,[1] Mossakowski,[2] and Pandalai and Patel.[3] Extending the analysis to include annular plates requires additional boundary conditions at the inner edge and thereby increases the mathematical complexity of the governing equation. In each of the cited papers, the critical buckling loads were determined from characteristic equations obtained from series solutions of the governing deflection equation. Including the inner-boundary conditions prohibits a series solution. Therefore, this Note employs finite-difference equations and the Vianello-Stodola iterative procedure to solve the title problem for several boundary conditions.

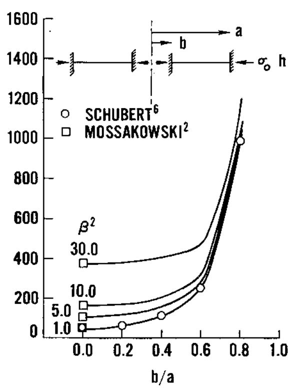

Fig. 1 Critical buckling loads, buckling parameter $\lambda = -\sigma_0 h a^2/Dr$ vs b/a, both edges fixed and loaded.

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Analysis

The governing axisymmetric equation in terms of the deflection w and the stresses σ_r and σ_θ is

$$\Delta w = \frac{h}{D_r}\left(\sigma_r \frac{d^2w}{dr^2} + \sigma_\theta \frac{1}{r}\frac{dw}{dr}\right) \qquad (1)$$

where

$$\Delta = \frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} - \frac{\beta^2}{r^2}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)$$

r,θ = radial and circumferential coordinates, respectively; $D_r = E_r h^3/12(1 - \nu_{r\theta}\nu_{\theta r})$; $\beta^2 = E_\theta/E_r$; E_θ, E_r = moduli of elasticity in the circumferential and radial directions, respectively; $\nu_{r\theta}, \nu_{\theta r}$ = Poisson's ratios, and h = plate thickness.

The stresses can be derived independently of the deflection from the compatibility equation, the stress-strain relations, and the equilibrium equation. Thus, the equations to determine the stresses are given by

$$d^2\sigma_r/dr^2 + (3/r)(d\sigma_r/dr) + (1 - \beta)/r^2\,\sigma_r = 0 \qquad (2)$$

and

$$d/dr(r\sigma_r) = \sigma$$

Integrating Eq. (2) gives

$$\sigma_r = C_1 r^{(\beta-1)} + C_2 r^{-(\beta+1)}$$

where C_1 and C_2 are determined from the loading conditions at the inner edge ($r = b$) and the outer edge ($r = a$). Substituting the stresses into Eq. (1) and writing the resulting equation in nondimensional form[4] gives

$$\nabla w + \lambda\left[(C_3\rho^{\beta-1} + C_4\rho^{-\beta-1})\frac{d^2w}{d\rho^2} + \frac{\beta}{\rho} \times (C_3\rho^{\beta-1} - C_4\rho^{-(\beta+1)})\frac{dw}{d\rho}\right] = 0 \qquad (3)$$

where:

$\nabla = \Delta$ with ρ substituted for r $\qquad C_3 = C_1 a^{\beta-1}/\sigma_0$

$\lambda = -\sigma_0 h a^2/D_r$ $\qquad C_4 = C_2 a^{-(\beta+1)}/\sigma_0$

$\rho = r/a$

Nontrivial solutions of Eq. (3) exist for particular values of the eigenvalue λ. The first eigenvalue determines the lowest critical buckling load σ_0.

Method of Solution

Equation (3) can be reduced to a second-order equation in terms of slope plus a constant of integration. For ease in

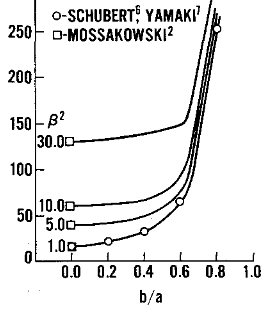

Fig. 2 Critical buckling loads, buckling parameter $\lambda = -\sigma_0 h a^2/Dr$ vs b/a, both edges simply-supported and loaded.